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GRAPHS WHOSE AUTOMORPHISM GROUPS CONTAIN THE ALTERNATING GROUPS

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ABSTRACT
There have been many investigations on the combinatorial structures and invariants over the group actions on the subsets of its elements. Automorphism groups from Graphs containing the cyclic and dihedral groups, C_n and D_n respectively have been constructed using Schur's Algorithm. In this paper, we seek to extend the work to graphs whose automorphism groups contain the Alternating group A_n. We plan to construct the graphs whose automorphism groups contain the Alternating group A_n. To construct these graphs, we will engage Schur's algorithm. We will first consider cases for $n = 3, 4, And\ 5$, which will in turn help us make generalizations for any value of n, the degree of the group in study. The process will involve, establishing the transitivity of the Alternating group A_n first, finding the adjacency matrices and then constructing the said graphs. We then determine a formula for calculating the number of graphs whose automorphism groups contain the Alternating group A_n. Through this study we have yielded important concepts and results in the field of group theory. We present the results of our findings from our workings as lemmas, graphs, and or tables where applicable.

INTRODUCTION
Here we focus on necessary definitions and theorems that are useful in the course of our project.
Group actions of different groups have resulted in various properties. G (a group) partitions a given X (a set it acts upon), it resulting in subsets referred to as orbits. Overtime, the orbits' numbers have been counted significantly using the Cauchy- Frobenius Lemma;

$| Orb_G(x) | = |G: Stab_G(x) |$.

A Mathematician by the name Schur came up with an algorithm we may use to determine the graphs whose groups of automorphism contain a transitive group G. It is this algorithm we intend to review and employ in our project.

Definition 1: In S_n, we may form a group of order $\frac{n!}{2}$ known as the alternating group from the set of all permutations that are even.

Definition 2: G (A group) acts on X(a set) transitively if there's only one orbit on X. Equivalently, it acts transitively if $\forall\ x, y \in X\ \exists\ g \in G\ s.t\ gx = y$.

Theorem 1: For G, a group acting on $X(a\ set)$ and $x \in X$, $|Orb_G(x)| = |G: Stab_G(x)|$,

Definition 3: Suppose G is a transitive group on X, G_x the stabilizer of the point $x \in X$. We refer to orbits $\Delta_0 = \{x\}$, Δ_1, Δ_2, …, Δ_{k-1} of G_x on X as sub-orbits of G.

Theorem 2: For any G, a group which acts on a finite set X, the cardinality of G- orbits is given by $\frac{1}{|G|}\sum_{g\in G}|Fix\ (g)|$ in X

Definition 4
Assume G acts on X transitively; Δ is orbit of G_x on a set X .
Let $\Delta^* = \{gx|\ g \in G, x \in g\Delta\}$, then Δ^* is also known as the G_x-orbit (or G -sub orbit) paired with Δ .

$\Delta^{**} = \Delta$ and $|\Delta| = |\Delta^*|$. Δ is self-paired If $\Delta = \Delta^*$.
G_x^{Δ} (The transitive constituent of G on Δ) forms a permutation group obtained when you restrict the elements of G_x to Δ.

Definition 5: $(g) = |Fix(g)|\ \forall\, g \in G$, defines Character (π) for the permutation representation of a group G on a set X.

Theorem 3: Consider definition 5, cardinality of self-paired sub-orbits of G is calculated as follows:

$$n_\pi = \frac{1}{|G|} \sum_{g\in G} \pi(g^2)\,, g \in G. \quad [5]$$

Definition 6: An n x n matrix gained from permuting the columns of an n x n identity matrix I_n is referred to as a permutation matrix.

Definition 7: Suppose V is a set of points known as vertices while E is set of vertices in twos not in any definite order (edges). A diagram with the sets V and E is called a graph denoted $\mathcal{G}(V, E)$ or sometimes $\mathcal{G}$(given no uncertainty on V and or E) .

Definition 8: Suppose $\mathcal{G}$ is a graph. A permutation α of the vertex set of $\mathcal{G}$, $\mathcal{V}(\mathcal{G})$ is an automorphism of $\mathcal{G}$ if $\forall$ $u, v \in \mathcal{V}(\mathcal{G})$;
$\{u, v\} \in E(\mathcal{G})$ iff $\{\alpha(u), \alpha(v)\} \in E(\mathcal{G})$

Definition 9: Automorphism group of $\mathcal{G}$ denoted Aut $(\mathcal{G})$ is all automorphisms' set in graph $\mathcal{G}$, considering the compositions (of functions). Ideally, it forms a sub-group of S_n on $\mathcal{V}(\mathcal{G})$.

Definition 10: Given $\mathcal{G}$ as a graph with n vertices labeled 1 up to n, an adjacency matrix $A(\mathcal{G})$ is the matrix defined by $A(\mathcal{G}) = (a_{ij}), i, j = 1, 2, 3, \ldots, n$ s.t;

$$a_{ij} = \begin{cases} 1; & if\ \exists\ an\ edge\ between\ the\ vertices\ i\ and\ j \\ 0; & otherwise \end{cases}$$

Definition 11: An n x n matrix whose entries is +1 or -1 and with mutually orthogonal rows is a Hadamard matrix.

MATERIALS AND METHODS

Literature Review

"Sabidussi [12]", found that for connected graphs their products is also connected and for a disconnected graph, the product thereof with any graph is disconnected while studying graph products. "Bouwer [3]" shows that if G is any given permutation group that's finite then exists infinitely several undirected and directed graphs which aren't isomorphic and whose groups of automorphisms has G as it's sub direct component.

A classification of all groups of permutations G having a sub orbit Δ length 4 where we have that $G_x \cong A_4$ or $G_x \cong S_4$ is faithful was done by Quirin, [11]. Leon [9] describes an algorithm for computing the automorphism group of a Hadamard matrix. He shows how to modify the algorithm for determining the equivalence of any two Hadamard matrices.

The algorithm yields the order of the automorphism group, the orbits of the automorphism groups on the rows and columns of the matrix and a set of permutations generating the automorphism group. Servatius, [13] on his study of graph groups, improves on a result by other Mathematicians who had looked into graph algebras with a finding that two graph algebras are isomorphic if and only if their graphs are isomorphic.

Babai *et al.*, [2] built a framework useful for studying the minimum number of edge orbits and showed that a bounded total number of edge-orbits are admitted as a representation of large classes of groups. In this case, if the group of automorphisms of X is isomorphic to G then a graph X is said to represent the group G.

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Cameron, [6] does a survey on finite graphs' automorphisms, especially the symmetry of the typical graphs. He dealt mainly with identifying automorphism groups as either abstract or permutation groups. He comes up a number of key findings among them being that a graph and its complement have the same automorphism group. He further discusses a finding by Frucht, [8] that all groups are also groups' automorphisms of a graph.

Chao, [7] used an algorithm developed from Schur's theorem to determine the graphs whose automorphism groups contain transitive groups. Olum, [10] used the concept to determine a formula for finding the tally of graphs whose automorphism groups contain given finite cyclic and dihedral groups. We expand this to alternating groups.

Schur's Algorithm Reflection

Schur's algorithm is constituted as below;

i. Consider a transitive permutation group G acting on n elements say {1, 2, 3… n} and G_1 the stabilizer of 1, then the orbits of G_1 are given as;
$\Delta_1 = \{1\}, \Delta_2, \Delta_3, \ldots, \Delta_k$.

ii. Associate each Δ_m with an n x n matrix as
$B(\Delta_m) = (b_{ij}), i, j = 1, 2, 3, \ldots, n$ s.t;
$$b_{ij} = \begin{cases} 1; & if\ \exists\ a\ g \in G\ and\ x \in \Delta_m\ where\ g\ 1 = j\ and\ gx = i \\ 0; & otherwise \end{cases}$$

iii. $B(\Delta_m)$ Is a symmetric matrix iff Δ_m is self-paired. The Identity matrix, I_n is clearly observed as $B(\Delta_1)$, hence we only have to find $B(\Delta_i)$ for $i = 2, 3, \ldots, k$.

iv. Consider each $B(\Delta_i)$ for $i = 2, 3, \ldots, k$ separately. If $B(\Delta_i)$ is a symmetric matrix, a graph X_i can be constructed whose adjacency matrix is given by $A(X_i) = B(\Delta_i)$.

v. We ignore $B(\Delta_i)$ for a moment if asymmetric.

vi. Now, proceed to sum $B(\Delta_i) + B(\Delta_j), i \neq j, i, j = 2, 3, \ldots, k$.

vii. Construct the graph for the sum if symmetric and ignore briefly if asymmetric.

viii. We then repeat the process of addition to find all the possible sums for 3, 4… k-1 different $B(\Delta_i)$ matrices.

ix. We use every symmetric matrix from the results as adjacency matrices to construct respective graphs.

x. Finally, the null graphs are constructed with n vertices.
This process gives all the automorphism groups from Graphs containing the transitive group G

RESULTS AND DISCUSSION

Transitivity of the alternating, A_n group

Let $\mathcal{G}$ be an alternating group A_n acting on a set $X = \{1, 2, 3, \ldots, n\}$.

Given $g \in \mathcal{G}$ and $x \in X$, each element $x \in X$ is fixed by exactly $\frac{|\mathcal{G}|}{n} = \frac{(n-1)!}{2}$.

$\Rightarrow |\mathrm{Stab}_G\, x| = \frac{(n-1)!}{2}\ \forall\ x \in \mathcal{X}$,

$\Rightarrow \sum_{x \in X} |\mathrm{StabG}\, x| = \frac{n!}{2}$.

$|\mathcal{G}| = \frac{n!}{2}$, since $\mathcal{G}$ is an alternating group of degree n.

$\Rightarrow |\mathrm{orb}_G\, x| = 1$.

$\Rightarrow \mathcal{G}$ acts transitively X.

Formulae:

Lemma 1:

Suppose $\mathcal{G}$ is the alternating group of degree n, the number of regular graphs whose groups of automorphisms contain $\mathcal{G}$ is 2; i.e

N ($\mathcal{G}$) = 2 (1)

Proof

In general, the Stabilizer of 1 is given by:

$\mathcal{G}_1$ = { 1, (234), (235),…….., (n-2 n n-1), … , (n-3 n)(n-2 n-1)} such that ;

$|\mathcal{G}_1| = \frac{(n-1)!}{2}$

For $n = 3$, The orbits of $\mathcal{G}_1$ are: $\Delta_1 = \{1\}$, $\Delta_2 = \{2\}$, $\Delta_3 = \{3\}$

Hence, Using Schur's algorithm, we have that;

$$B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Setting A (X_1) = B (Δ_1) and A (X_2) = B (Δ_2) + B (Δ_3) as adjacency matrices for the graphs X_1 and X_2 respectively.

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) + B(\Delta_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Two graphs may be constructed with X_1 and X_2 being a null graph and a complete graph respectively of 3 vertices.

Figure:

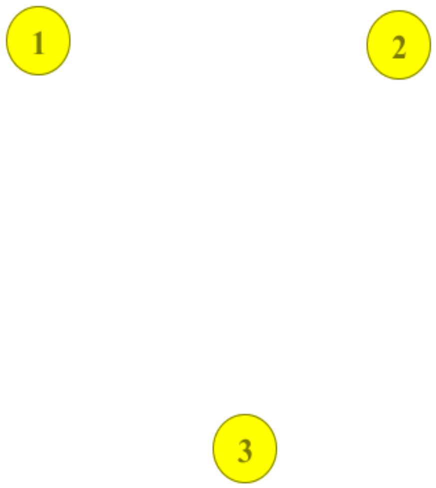

Figure 1 (a) X_1

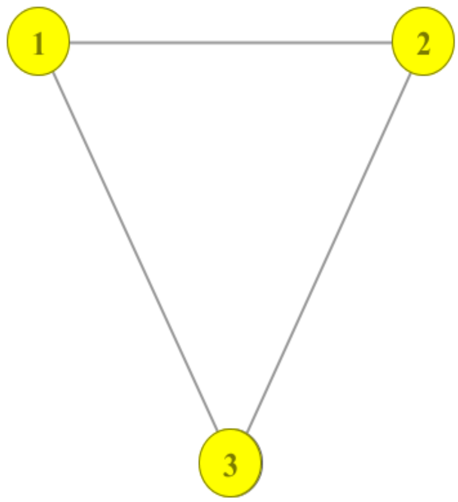

Figure 1 (b) X_2

For $n > 3$, The orbits of $\mathcal{G}_1$ are $\Delta_1 = \{1\}$, $\Delta_2 = \{2, 3, 4, \dots, n\}$

Hence, Using Schur's algorithm, we establish B (Δ_2), as below;

$$B(\Delta_2) = \begin{pmatrix} 0\ 1\ 1\ 1 \dots \dots \dots 1 \\ 1\ 0\ 1\ 1 \dots \dots \dots 1 \\ 1\ 1\ 0\ 1 \dots \dots \dots 1 \\ 1\ 1\ 1\ 0 \dots \dots \dots 1 \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ 1\ 1\ 1\ 1 \dots \dots 0\ 1 \\ 1\ 1\ 1\ 1 \dots \dots \dots 0 \end{pmatrix}$$

Setting A (X_1) = B (Δ_1) and A (X_2) = B (Δ_2) as adjacency matrices for the graphs X_1 and X_2 respectively;

$$A(X_1) = B(\Delta_1) = \begin{pmatrix} 1\ 0\ 0\ 0 \dots \dots \dots 0 \\ 0\ 1\ 0\ 0 \dots \dots \dots 0 \\ 0\ 0\ 1\ 0 \dots \dots \dots 0 \\ 0\ 0\ 0\ 1 \dots \dots \dots 0 \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ 0\ 0\ 0\ 0 \dots \dots 1\ 0 \\ 0\ 0\ 0\ 0 \dots \dots \dots 1 \end{pmatrix}$$

$$A(X_2) = B(\Delta_2) = \begin{pmatrix} 0\ 1\ 1\ 1 \dots \dots \dots 1 \\ 1\ 0\ 1\ 1 \dots \dots \dots 1 \\ 1\ 1\ 0\ 1 \dots \dots \dots 1 \\ 1\ 1\ 1\ 0 \dots \dots \dots 1 \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ 1\ 1\ 1\ 1 \dots \dots 0\ 1 \\ 1\ 1\ 1\ 1 \dots \dots \dots 0 \end{pmatrix}$$

Two graphs may be constructed with X_1 and X_2 being a null graph and a complete graph respectively of n vertices.

■

Alternative Proof for $n > 3$

In general, the Stabilizer of 1 is given by:

$\mathcal{G}_1 = \{$ 1, (234), (235),…….., (n-2 n n-1), … , (n-3 n)(n-2 n-1)$\}$ such that ;

$|\, \mathcal{G}_1 \,| = \frac{(n-1)!}{2}$

For $n > 3$, the orbits of $\mathcal{G}_1$ are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4, \dots , n\}$

$\Longrightarrow$ There are 2 orbits for the stabilizer of 1.

Clearly, the 2 orbits of $\mathcal{G}_1$ are self-paired by definition 4.

This implies that the matrices associated with the sub-orbits of $\mathcal{G}$ are symmetric.

But the number of graphs whose groups of automorphisms contain $\mathcal{G}$ is equal to the total number of symmetric matrices by Schur's algorithm, then;

The number of graphs whose groups of automorphisms contain $\mathcal{G}$ = 2.

The identity matrix yields the null graph while the other symmetric matrix yields a complete graph

■

Example

Taking $\mathcal{G} = A_{15}$, find the number of regular graphs whose groups of automorphism contains $\mathcal{G}$ and construct the graphs.

Solution

Follows from the above lemma 1;

The orbits of A_9 are $\Delta_1 = \{ 1 \}$, $\Delta_2 = \{ 2, 3, 4,5,6,7,8,9,10,11,12,13,14,15\}$

The 2 orbits are self- paired and hence they form symmetric matrices.

The 2 symmetric matrices may be used to form 2 adjacency matrices for our graphs

$\Longrightarrow$ There are 2 graphs whose groups of automorphisms contains A_{15} .

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Figure:

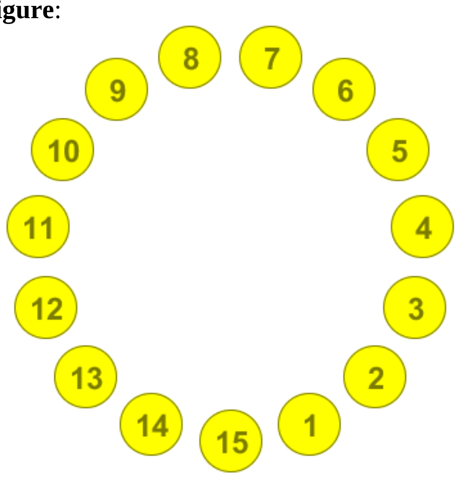

Figure 2 (a) X_1

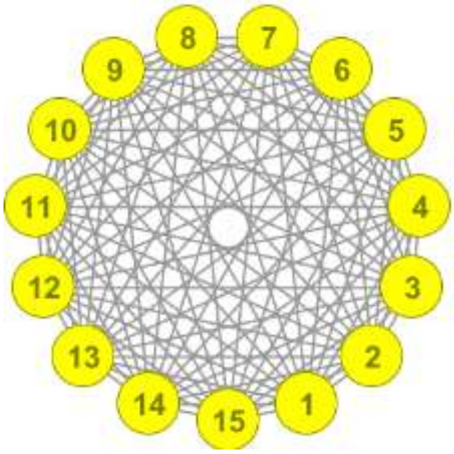

Figure 2 (b) X_2

CONCLUSION

We have employed Schur's algorithm in this project to construct graphs whose groups of automorphisms contains the Alternating groups. We have as well calculated the number of graphs whose automorphism groups contains the Alternating group.

We have shown our results as above and given examples of constructed graphs in Figure 1 & 2. The main result from our study has been expressed in lemma 1. Lemma 1 shows that in determining the number of graphs whose automorphism groups contain the Alternating group, the answer is always 2 being the null and complete graphs. Obviously the 2 graphs represent the Symmetric group.

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